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Reduced Effort Control Laws for Underactuated Rigid Spacecraft

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A nonsmooth, time-invariant feedback control law can be used to rotate an axisymmetric rigid spacecraft to the zero equilibrium using only two control torques. This method, however, may require a significant amount of control effort, especially for initial conditions close to an equilibrium manifold corresponding to rotations about the unactuated principal axis. A control law is proposed in this work that reduces the control effort required to perform rest-to-rest maneuvers for initial conditions close to this equilibrium manifold. Specifically, the phase space of the system is divided into two parts, one corresponding to initial conditions producing large control effort (the "bad" region) and the other corresponding to initial conditions producing small control signals (the "good" region). The proposed control law then renders this undesirable equilibrium manifold unstable, driving the trajectories of the closed-loop system into the good region, where the original control law is subsequently used. Numerical simulations indicate reduction of the control magnitude on the order of 80–90% for initial conditions close to the equilibrium manifold.

I. Introduction

THE problem of stabilization of a rigid body using less than three control inputs has received considerable attention in the recent literature. Both the problems of the stabilization of the dynamics and the stabilization of the kinematics have been treated in the literature.[1–6] The stabilization problem of the complete system, i.e., the dynamics and the kinematics, has been addressed in Refs. 7–13. The attitude stabilization of an axially symmetric rigid body using two independent control torques was studied by Krishnan et al.[8,9] and Tsiotras et al.[10] If the uncontrolled principal axis is not the axis of symmetry, the system is strongly accessible and small time locally controllable.[9] When the uncontrolled axis coincides with the axis of symmetry, the complete system fails to be controllable or even accessible. However, the system equations are strongly accessible and small time locally controllable in the case of zero spin rate. A nonlinear control approach was developed in Ref. 8, which achieves arbitrary reorientation for this restricted case. In Refs. 14 and 15, the authors presented a new formulation of the attitude kinematics, which was used in Ref. 10 to solve the same problem avoiding the successive switchings of Ref. 8. References 8 and 10 treated the axisymmetric case, whereas the nonsymmetric case was dealt with in Refs. 11–13 and 16.

In this paper, a modification of the control law presented in Ref. 10 for the attitude stabilization of an axisymmetric rigid body using two independent control torques is proposed. Because Brockett's necessary condition for smooth stabilizability is not satisfied for this system, any stabilizing (time-invariant) control law is necessarily nonsmooth. (Stabilizing time-varying smooth control laws, however, may still exist.) This nonsmoothness is evident in Ref. 10 in the form of the nondifferentiability of the control law at the origin. Because of the singularity at the origin, this control law may saturate the actuators, especially for initial conditions close to the equilibrium manifold. Therefore, it is desirable to modify the control law of Ref. 10 to reduce the required control signals. Compared to the control law in Ref. 10, the modified control law proposed here remedies this large control input problem by driving the trajectories of the closed-loop system away from the singular equilibrium manifold, toward a region in the state space where the high-authority part of the control input remains small and bounded. The procedure is simple and can be easily validated using phase portrait considerations. A numerical example illustrates the control effort improvement using the new control law.

II. Underactuated Spacecraft

The dynamics of a rigid spacecraft with two controls can be written as

$$\dot{\omega}_1 = a_1\omega_2\omega_3 + u_1 \tag{1a}$$

$$\dot{\omega}_2 = a_2\omega_3\omega_1 + u_2 \tag{1b}$$

$$\dot{\omega}_3 = a_3\omega_1\omega_2 \tag{1c}$$

where a_i are the inertia parameters satisfying $a_1 + a_2 + a_3 + a_1a_2a_3 = 0$. Here we assume a body-fixed reference frame along the principal axes of inertia.

Equations (1) describe an underactuated spacecraft with no control authority about the third principal axis. Notice that, in this case, ω_3 can be controlled only indirectly through judicious choice of the time histories of $\omega_1(t)$ and $\omega_2(t)$. In case of an axisymmetric body (about the 3-axis), $a_3 = 0$ and $a_1 = -a_2 = a$, and Eqs. (1) reduce to

$$\dot{\omega}_1 = a\omega_{30}\omega_2 + u_1 \tag{2a}$$

$$\dot{\omega}_2 = -a\omega_{30}\omega_1 + u_2 \tag{2b}$$

$$\dot{\omega}_3 = 0 \tag{2c}$$

where $\omega_3(0) = \omega_{30}$ is constant. Introducing the complex variables $\omega = \omega_1 + i\omega_2$ and $u = u_1 + iu_2$ (with $i = \sqrt{-1}$), the preceding equations can be written as

$$\dot{\omega} = -ia\omega_{30}\omega + u \tag{3}$$

III. Kinematics of the Attitude Motion

The orientation of a rigid spacecraft can be specified using various parameterizations, for example, Eulerian angles, Euler parameters, Cayley–Rodrigues parameters, Cayley–Klein parameters, etc.[17] Recently, a new parameterization using a pair of complex and real coordinates was introduced.[14,15] According to these results, the relative orientation between two given reference frames can be represented by two rotations, one corresponding to the real coordinate z and the other corresponding to the complex coordinate w. Specifically, one can align an (inertial) reference frame to a body-fixed frame by first performing an initial rotation of magnitude z about, e.g., the inertial 3-axis and then performing a second rotation to move the intermediate 3-axis to the body 3-axis. The second rotation can be

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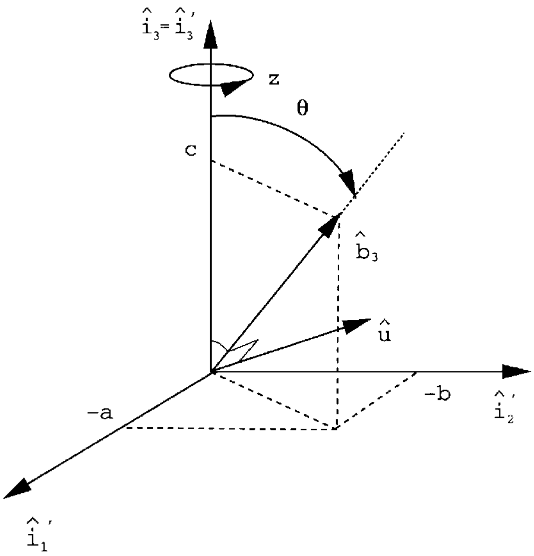

Fig. 1 Attitude representation using (w, z) coordinates.

completely characterized by the complex coordinate $w = w_1 + i w_2$. It is a rotation of magnitude

$$\theta = \arccos\left(\frac{1 - |w|^2}{1 + |w|^2}\right) \tag{4}$$

about the unit vector

$$\hat{\boldsymbol{u}} = \left(\frac{w + \bar{w}}{2|w|}\right)\hat{\boldsymbol{i}}_1' + \left(\frac{i(\bar{w} - w)}{2|w|}\right)\hat{\boldsymbol{i}}_2' \tag{5}$$

This situation is depicted in Fig. 1, where $(\hat{\boldsymbol{i}}_1', \hat{\boldsymbol{i}}_2', \hat{\boldsymbol{i}}_3')$ is the intermediate reference frame resulting from the rotation z about the inertial $\hat{\boldsymbol{i}}_3$ axis and where (a, b, c) denote the coordinates of the unit vector along the $\hat{\boldsymbol{i}}_3'$ axis in the body frame, that is,

$$\hat{\boldsymbol{i}}_3' = a\hat{\boldsymbol{b}}_1 + b\hat{\boldsymbol{b}}_2 + c\hat{\boldsymbol{b}}_3 \tag{6}$$

It can be shown that the coordinates of the $\hat{\boldsymbol{b}}_3$ axis in the $\hat{\boldsymbol{i}}'$ frame are also related to (a, b, c)[15]:

$$\hat{\boldsymbol{b}}_3 = -a\hat{\boldsymbol{i}}_1' - b\hat{\boldsymbol{i}}_2' + c\hat{\boldsymbol{i}}_3' \tag{7}$$

With this notation, w represents the stereographic coordinates corresponding to the unit vector (a, b, c) defined by[15,18]

$$w = (b - ia)/(1 + c) \tag{8}$$

Alternatively, the equations

$$a = \frac{i(w - \bar{w})}{1 + |w|^2}, \qquad b = \frac{w + \bar{w}}{1 + |w|^2}, \qquad c = \frac{1 - |w|^2}{1 + |w|^2} \tag{9}$$

and can be used to find a, b, and c once w is known. Here $|\cdot|$ denotes the absolute value of a complex number, i.e., $w\bar{w} = |w|^2$, $w \in \mathbb{C}$.

The kinematic equations, which provide the geometric constraints of the motion and relate the rates of the kinematic parameters w and z to the angular velocity vector, can be written as follows[10,15]:

$$\dot{w} = -i\omega_3 w + (\omega/2) + (\bar{\omega}/2)w^2 \tag{10a}$$

$$\dot{z} = \omega_3 + \mathrm{Im}(\omega\bar{w}) \tag{10b}$$

Notice that these equations take the convenient form

$$\frac{\mathrm{d}}{\mathrm{d}t}|w|^2 = (1 + |w|^2)\,\mathrm{Re}(\omega\bar{w}) \tag{11a}$$

$$\dot{z} = \omega_3 + \mathrm{Im}(\omega\bar{w}) \tag{11b}$$

where the bar denotes complex conjugate and Re( ) and Im( ) denote the real and imaginary parts of a complex number, respectively. In Eq. (11b) only the imaginary part of the product $\omega\bar{w}$ appears, whereas in Eq. (11a) only the real part appears. This duality (or antisymmetry) of Eqs. (11a) and (11b) is desirable and can be used to derive stabilizing control laws for the kinematics described by Eqs. (10). Clearly, $w = 0$ if and only if $|w| = 0$, and stabilization of the system in Eqs. (10) is equivalent to stabilization of the system in Eqs. (11). References 10, 18, and 19 indicate that the coordinates (w, z) offer some significant advantages for attitude analysis and control problems.

IV. Problem Statement

Consider an axisymmetric body with the applied torque vector in the plane that is perpendicular to the symmetry axis. In such a case, the system is described by Eqs. (2) and, thus, ω_3 remains constant. If initially $\omega_3(0) \neq 0$, no control input can bring the system to the equilibrium. The system is not controllable to the equilibrium, but it is controllable to the submanifold $\omega = w = 0$ in the (ω, ω_3, w, z) space. For a more detailed discussion on this issue, refer to Refs. 8–10. Therefore, for an axisymmetric body, actively controlled rotation to the equilibrium for the system in Eqs. (3–10) makes sense only if $\omega_3 \equiv 0$. In this case, the system equations simplify to

$$\dot{\omega} = u \tag{12a}$$

$$\dot{w} = (\omega/2) + (\bar{\omega}/2)w^2 \tag{12b}$$

$$\dot{z} = \mathrm{Im}(\omega\bar{w}) \tag{12c}$$

This system can be stabilized to the origin, but any time-invariant stabilizing control law has to be necessarily nonsmooth, inasmuch as Eqs. (12) fail Brockett's necessary condition for smooth stabilizability.[20] Therefore, we concentrate on using nonsmooth (albeit time-invariant) stabilizers for this system.

Equations (12) represent a system in cascade form, with the kinematics (12b) and (12c) being the driven subsystem and the dynamics (12a) being the driving subsystem. The methodology in Ref. 10 used this structure to derive a nonsmooth control law to stabilize Eqs. (12). In essence, the controller design consists of a two-step process. In the first step, only stabilization of the kinematics is addressed, with the angular velocity treated as the control input. In the second step, the control torque u is chosen to shape the desired velocity profile. Because the angular velocity in the first step is (necessarily) a nonsmooth function of w and z, caution should be exercised when implementing this angular velocity in the second step. The nonsmooth controller of Ref. 10, along with its potential drawbacks, is summarized in the next section.

V. Nonsmooth Controller for the Kinematics

In Ref. 10 a nonsmooth control law was proposed for the kinematic system described by

$$\dot{w} = (\omega/2) + (\bar{\omega}/2)w^2 \tag{13a}$$

$$\dot{z} = \mathrm{Im}(\omega\bar{w}) \tag{13b}$$

and was later implemented through the integrator in Eq. (12a). The proposed control law in Ref. 10 was motivated by the decoupling of these equations with respect to the product $\omega\bar{w}$, as evident from the discussion following Eqs. (11). This control law is given by

$$\omega = -\kappa w - i\mu(z/\bar{w}) \tag{14}$$

where $\mu > \kappa/2 > 0$. With this control law, the closed-loop system in terms of $|w|$ and z is given by

$$\frac{\mathrm{d}|w|^2}{\mathrm{d}t} = -\kappa(1 + |w|^2)|w|^2 \tag{15a}$$

$$\dot{z} = -\mu z \tag{15b}$$

which is globally exponentially stable. As can be easily inferred by observing Eqs. (14) and (11), the first term in the control law (14) has an effect only on the differential equation for w, whereas the second term in Eq. (14) has an effect only on the differential equation for

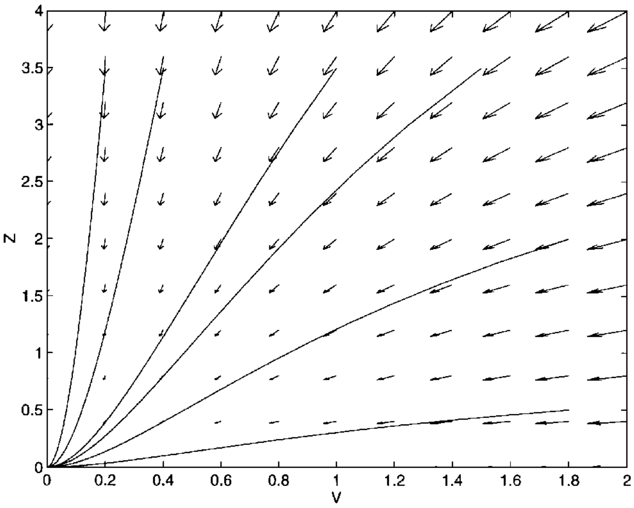

Fig. 2 Phase portrait of system in Eqs. (16).

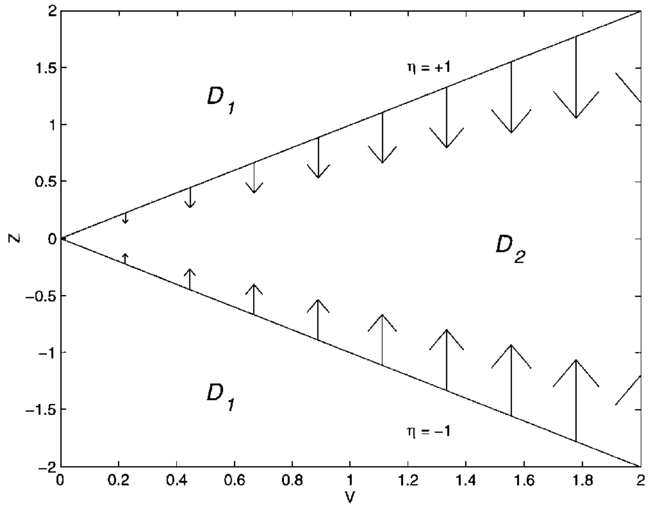

Fig. 3 Regions $\mathcal{D}_1$ and $\mathcal{D}_2$ in (z, v) phase space.

z. Moreover, the second term in Eq. (14) is a nonsmooth function of w.

The main disadvantage of the control law in Eq. (14) is that the last term, which involves the ratio $z/\bar{w}$, may become unbounded without careful choice of the gains. The previously imposed gain condition $\mu > \kappa/2$ ensures that the rate of decay of z is at least as large as the rate of decay of w, such that their ratio remains bounded. Actually, one can easily establish from Eqs. (15) that, for $\mu > \kappa/2$, along the solutions of the system, $z/\bar{w} \to 0$ as $t \to \infty$.

Introducing the variable $v = |w|^2$, the system in Eqs. (15) takes the form

$$\dot{v} = -\kappa(1 + v)v \tag{16a}$$

$$\dot{z} = -\mu z \tag{16b}$$

This is a system that evolves on $\mathbb{R}_+ \times \mathbb{R}$. Typical trajectories and the vector field of the closed-loop system in Eqs. (16) for $\kappa = 1$ and $\mu = 2$ are shown in Fig. 2. (Because z does not change sign, it suffices to plot only the $z > 0$ case.)

Although in Eq. (14) the ratio $z/\bar{w}$, and hence the control effort ω, remains bounded by proper choice of control gains, the control input ω may take large values in the region where w is small. From Eq. (15a), $|w(t)| \le |w(0)|$ for all $t \ge 0$, and for small initial conditions $w(0)$, the control law may use a substantial amount of energy, especially in regions where $|z|$ is large. In Fig. 2, for example, the region that is close to the z axis is clearly undesirable as far as control expenditure is concerned. Modification of the control law in Eq. (14), such that the vector field close to the z axis points away from this axis, is highly desirable. In short, the idea is to divide the (z, v) phase space into two regions according to the value of the ratio

$$\eta = z/|w|^2 = z/v \tag{17}$$

This ratio is a direct indication of the relative magnitude between z and w. This ratio should be kept small to avoid high control effort. Hence, if initially the states are in an undesirable region where η attains large values, the feedback control strategy should drive the trajectories to a safe region in the state space where η remains relatively small. Without loss of generality, choose as undesirable the region where $|\eta| > 1$, leaving $|\eta| \le 1$ as the desirable region. Therefore, these two regions, denoted by $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively, are defined by

$$\mathcal{D}_1 = \{(z, v) \in \mathbb{R} \times \mathbb{R}_+ : \infty > |\eta| > 1\} \tag{18a}$$

$$\mathcal{D}_2 = \{(z, v) \in \mathbb{R} \times \mathbb{R}_+ : |\eta| \le 1\} \tag{18b}$$

These two regions are shown in Fig. 3.

VI. Main Results

The proposed modification to the control law in Eq. (14) is simple. Positive feedback is used for v when the trajectory is in region $\mathcal{D}_1$, while z is decreasing. This change will make the manifold $v = 0$ (equivalently, $w = 0$) unstable, and the trajectories will move toward the region $\mathcal{D}_2$ and subsequently stay there. The control law in region $\mathcal{D}_2$ is essentially the same as in Eq. (14). Notice that, by definition, inside the region $\mathcal{D}_2$ we have $|\eta| \le 1$, and because $|z|/|\bar{w}| = |\eta||w|$ we can ensure that ω will not take excessive values as long as the trajectories remain in $\mathcal{D}_2$. These statements will be made more precise in the sequel.

A. Proposed Control Law for Kinematics

The proposed control law for the system in Eqs. (13) is defined by

$$\omega = -\kappa(\eta)w - i\mu(\eta)(z/\bar{w}) \tag{19}$$

where $\kappa(\eta)$ and $\mu(\eta)$ are smooth functions satisfying

$$-\kappa_c \le \kappa(\eta) < 0, \qquad 0 \le \mu(\eta) < (\mu_c/2) \qquad \forall (z, v) \in \mathcal{D}_1 \tag{20a}$$

$$0 \le \kappa(\eta) \le \kappa_c, \qquad \mu_c/2 \le \mu(\eta) \le \mu_c \qquad \forall (z, v) \in \mathcal{D}_2 \tag{20b}$$

and $0 < \kappa_c < \mu_c$. One possible choice is, for example,

$$\kappa(\eta) = (2\kappa_c/\pi)\arctan[\rho(1 - \eta^2)] \tag{21a}$$

$$\mu(\eta) = (\mu_c/\pi)\arctan[\rho(1 - \eta^2)] + (\mu_c/2) \tag{21b}$$

From Eqs. (21), κ and μ are bounded as

$$-\kappa_c \le \kappa(\eta) \le \kappa_c, \qquad 0 \le \mu(\eta) \le \mu_c \tag{22}$$

for all $\eta \in \mathbb{R}$. Moreover, notice that $\kappa(\eta) < 2\mu(\eta)$ for all $(z, v) \in \mathcal{D}_2$.

The next theorem gives the main result of the paper.

Theorem 1: Consider the system in Eqs. (13), and let the control law be as in Eqs. (19–21) with $0 < \kappa_c < \mu_c$. Then for initial conditions $[z(0), w(0)] \in \mathbb{R} \times (\mathbb{C}\backslash\{0\})$, the following properties hold.

1) Coordinate $w(t) \neq 0$, $\forall t \ge 0$.

2) The trajectory $[z(\cdot), w(\cdot)]$ is bounded and

$$\lim_{t\to\infty} [z(t), w(t)] = 0 \tag{23}$$

3) The control law $\omega(\cdot)$ is bounded, and it has a bounded derivative.

With the control law in Eq. (19), the closed-loop system takes the form

$$\dot{v} = -\kappa(\eta)(1 + v)v \tag{24a}$$

$$\dot{z} = -\mu(\eta)z \tag{24b}$$

where $v = |w|^2$ and η as in Eq. (17). From Eq. (22) we have that z decays monotonically for all initial conditions, whereas v increases in the region $\mathcal{D}_1$ and decreases in $\mathcal{D}_2$. The result is that the trajectories of Eqs. (24) tend to $\mathcal{D}_2$ and then to the origin, as required.

Before proving Theorem 1 we need to establish the following two lemmas.

Lemma 1: The region $\mathcal{D}_2$ is invariant for the system in Eqs. (24).

Proof: The boundary of the set $\mathcal{D}_2$ is given by the two lines $\eta = \pm 1$ (cf. Fig. 3). On the boundary of $\mathcal{D}_2$ the feedback gains are $\kappa(\eta) = 0$ and $\mu(\eta) = \mu_c/2$. The vector field on the boundary of $\mathcal{D}_2$ is, therefore,

$$\dot{v} = 0 \tag{25a}$$

$$\dot{z} = -(\mu_c/2)z \tag{25b}$$

which points into the interior of $\mathcal{D}_2$. Therefore, trajectories in $\mathcal{D}_2$ cannot escape this region, and thus it is invariant for the closed-loop system in Eqs. (24). □

This lemma establishes that, for initial conditions in $\mathcal{D}_2$, the trajectories of the closed-loop system remain in $\mathcal{D}_2$ for all times. Equivalently, if at some time $t' \geq 0$ the trajectory enters $\mathcal{D}_2$, it stays in $\mathcal{D}_2$ for all $t \geq t'$. Figure 3 shows the vector field on the boundary of $\mathcal{D}_2$.

Lemma 2: Consider the system in Eqs. (24). For all initial conditions $(z, v) \in \mathcal{D}_1$, the trajectories enter the region $\mathcal{D}_2$ in finite time.

Proof: As long as $(z, v) \in \mathcal{D}_1$, from Eq. (20a) μ is bounded as $0 \leq \mu(\eta) < \mu_c/2$. This implies that z is bounded. Actually, $|z(t)| \leq |z(0)|$ for all $t \geq 0$. Note that z does not change sign for all $t \geq 0$. Without loss of generality, assume that $z(0) \geq 0$ [the case $z(0) \leq 0$ being similar]. If $[z(0), v(0)] \in \mathcal{D}_1$ then, by definition $\eta(0) > 1$. The derivative of η in $\mathcal{D}_1$ is then

$$\begin{aligned}\dot{\eta} &= (\dot{z}/v) - (z/v^2)\dot{v} \\ &= -\mu(\eta)\eta + \kappa(\eta)(1+v)\eta \\ &\leq -\mu(\eta)\eta \leq 0\end{aligned} \tag{26}$$

because $\kappa(\eta) < 0$ and $v > 0$; hence, η is bounded in $\mathcal{D}_1$. Let $\mathrm{cl}\,\mathcal{D}_1$ denote the closure of $\mathcal{D}_1$ in $\mathbb{R}^2$, that is,

$$\begin{aligned}\mathrm{cl}\,\mathcal{D}_1 &= \mathcal{D}_1 \cup \{(z, v) \in \mathbb{R}\times\mathbb{R}_+ : |\eta| = 1\} \\ &\quad \cup \{(z, v) \in \mathbb{R}\times\mathbb{R}_+ : v = 0\}\end{aligned} \tag{27}$$

Then it is an easy exercise to show that $\dot{\eta} \neq 0$ for all $(z, v) \in \mathrm{cl}\,\mathcal{D}_1 \backslash \{(0,0)\}$. Hence, there exists $\epsilon > 0$ such that $\dot{\eta} < -\epsilon$ in $\mathcal{D}_1$ and, consequently, η monotonically decreases. Thus, every trajectory starting in $\mathcal{D}_1$ will leave this set and enter $\mathcal{D}_2$ in finite time. □

Notice that the set $\{(z, v) \in \mathbb{R}\times\mathbb{R}_+ : v = 0 \text{ and } z \neq 0\}$ is an unstable manifold for the closed-loop system. Figure 3 shows the vector field on the boundary of $\mathcal{D}_1$. The following corollary follows directly from Lemmas 1 and 2.

Corollary 1: Consider the system in Eqs. (24). For all initial conditions $[z(0), v(0)] \in \mathbb{R}\times(\mathbb{R}_+\backslash\{0\})$, η is bounded for all $t \geq 0$.

We are now ready to give the proof of Theorem 1.

Proof: From Eqs. (24a) and (22) we have that

$$\dot{v} \geq -\kappa_c(1+v)v \tag{28}$$

where $\kappa_c > 0$. The solution of the differential equation

$$\dot{x} = -\kappa_c(1+x)x, \qquad x(0) = x_0 > 0 \tag{29}$$

is given by

$$x(t) = \frac{1}{c_0 e^{\kappa_c t} - 1} \tag{30}$$

where $c_0 = (x_0 + 1)/x_0$. Clearly, $x(t) \neq 0$ for all $t \geq 0$ and $\lim_{t\to\infty} x(t) = 0$. Therefore, $v(\cdot)$ is bounded below by the solutions of the differential equation (29) subject to initial condition $x_0 = v(0)$. Hence, $|w(t)| \neq 0$ for all $t \geq 0$ and $w(\cdot)$ approaches the origin asymptotically.

We now show that $\lim_{t\to\infty}[z(t), v(t)] = 0$. If $[z(0), v(0)] \in \mathcal{D}_2$, then according to Lemma 1 we have that $[z(t), v(t)] \in \mathcal{D}_2$ for all $t \geq 0$ and $\mathcal{D}_2$ is an invariant set for the closed-loop system. Consider now the positive definite, radially unbounded function $V : \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}_+$ given by

$$V(z, v) = \tfrac{1}{2}v^2 + \tfrac{1}{2}z^2, \qquad \forall (z, v) \in \mathcal{D}_2 \tag{31}$$

The derivative of V along the trajectories of Eqs. (24) is

$$\dot{V} = -\kappa(\eta)(1+v)v^2 - \mu(\eta)z^2 \leq 0, \qquad \forall (z, v) \in \mathcal{D}_2 \tag{32}$$

therefore, the trajectories are bounded in $\mathcal{D}_2$. Moreover, $\dot{V} = 0$ if and only if $\kappa(\eta)(1+v)v^2 + \mu(\eta)z^2 = 0$. Using the definitions of $\kappa(\eta)$ and $\mu(\eta)$ in $\mathcal{D}_2$ and recalling that $v \geq 0$, one establishes that the last equality is not satisfied in $\mathcal{D}_2$ unless $z = v = 0$. By LaSalle's theorem, $\lim_{t\to\infty}[z(t), v(t)] = 0$ for all initial conditions in $\mathcal{D}_2$. To finish the proof, recall from Lemma 2 that, if $[z(0), v(0)] \in \mathcal{D}_1$, then $|z|$ is bounded by $|z(0)|$ and there exists a time $t' > 0$ such that $[z(t'), v(t')] \in \mathcal{D}_2$. This implies that, for all $t' \geq t \geq 0$, the trajectories in $\mathcal{D}_1$ are bounded and are confined inside the strip $|z(t)| \leq |z(0)|$. However, according to the preceding discussion, the trajectory with initial condition $[(z(t'), v(t')]$ satisfies $\lim_{t\to\infty}[z(t), v(t)] = 0$. Therefore, it has been shown that, for all $[z(0), v(0)] \in \mathbb{R}\times(\mathbb{R}_+\backslash\{0\})$, the trajectories remain bounded and have the property that $\lim_{t\to\infty}[z(t), v(t)] = 0$. By the definition of v, this implies that

$$\lim_{t\to\infty}[z(t), w(t)] = 0 \tag{33}$$

To show that ω is bounded, write the ratio $z/\bar{w} = \eta w$. From Eq. (19) one obtains

$$|\omega| \leq \kappa_c|w| + \mu_c|\eta||w| \tag{34}$$

From Corollary 1, for all initial conditions $[z(0), w(0)] \in \mathbb{R}\times(\mathbb{C}\backslash\{0\})$, η is bounded. Because w is also bounded, from Eq. (34) it follows that ω is bounded.

From Eq. (13a) it follows immediately that $\dot{w}$ is also bounded. Moreover, because

$$\dot{\eta} = -\mu(\eta)\eta + \kappa(\eta)(1+v)\eta \tag{35}$$

and $\mu(\eta)$, $\kappa(\eta)$, v, and η are all bounded, we have that $\dot{\eta}$ is bounded.

The derivative of ω is given by

$$\dot{\omega} = -\dot{\kappa}(\eta)w - \kappa(\eta)\dot{w} - i\dot{\mu}(\eta)\eta w - i\mu(\eta)\dot{\eta}w - i\mu(\eta)\eta\dot{w} \tag{36}$$

Using Eqs. (21) one has

$$\dot{\kappa}(\eta) = -\frac{4\kappa_c}{\pi}\frac{\rho}{1+\rho^2(1-\eta^2)^2}\eta\dot{\eta} \tag{37a}$$

$$\dot{\mu}(\eta) = -\frac{2\mu_c}{\pi}\frac{\rho}{1+\rho^2(1-\eta^2)^2}\eta\dot{\eta} \tag{37b}$$

Because $\dot{\eta}$ is bounded, $\dot{\kappa}(\eta)$ and $\dot{\mu}(\eta)$ are both bounded. Finally, the boundedness of $\dot{\omega}$ follows directly from Eq. (36) and the fact that all of the terms in the right-hand side of this equation are bounded. □

The vector field and the corresponding trajectories of the closed-loop system with the control law in Eq. (19) are shown in Fig. 4 (compare with Fig. 2).

Remark 1: Theorem 1 shows that, for all initial conditions $w(0) \neq 0$, the control law in Eq. (19) drives the system trajectories to the origin. This control law cannot be used if $w(0) = 0$ (and $z \neq 0$). Linearization of the system represented by Eq. (12) about $w = 0$, however, shows that this system is controllable, and choosing, for example, a constant control $\omega = \omega_c \in \mathbb{C}$, one can move away from the z axis into the $\mathcal{D}_1$ region; once in $\mathcal{D}_1$, use of the control in Eq. (19) drives the system to the origin.

Remark 2: Another choice of a feedback control for Eqs. (13) is the sublinear control in terms of w,

$$\omega = -\kappa\frac{w}{1+|w|^2} - i\mu\frac{z}{\bar{w}} \tag{38}$$

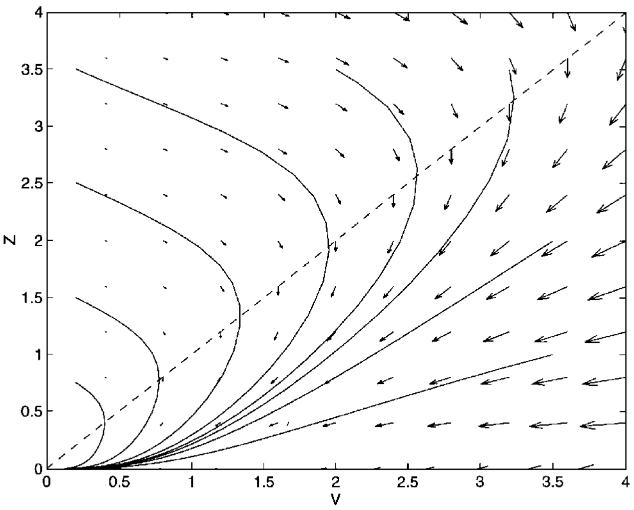

Fig. 4 Phase portrait of system in Eqs. (24).

which renders the closed-loop system

$$\dot{v} = -\kappa v \tag{39a}$$

$$\dot{z} = -\mu z \tag{39b}$$

globally exponentially stable. The earlier methodology can be applied mutatis mutandis to this control law, as well. Moreover, several other similar modifications can be introduced to the control law in Eq. (14). It should be evident that the results in this section can be applied to these control laws with only minor modifications.

B. Proposed Control Law for Complete System

The control law in Eq. (19) was shown to achieve $\lim_{t\to\infty}[z(t), w(t)] = 0$. Moreover, it is a bounded controller with bounded derivative. This allows one to implement this control through the dynamics in Eq. (12a). To this end, define the error

$$e = \omega - \omega_d \tag{40}$$

where ω_d is the desired angular velocity profile given in Eq. (19). Consider the feedback control

$$u = \dot{\omega}_d - \alpha[\omega + \kappa(\eta)w + i\mu(\eta)\eta w] \tag{41}$$

where $\alpha > 0$ and where $\dot{\omega}_d$ is given in Eq. (36), along with Eqs. (37). The value of $\dot{\eta}$ is now given by

$$\dot{\eta} = -\mu(\eta)\eta + \kappa(\eta)(1 + v)\eta + \mathrm{Im}(e/w) - (1 + v)\eta\,\mathrm{Re}(e/w) \tag{42}$$

With the control law in Eq. (41), the closed-loop system takes the form

$$\dot{e} = -\alpha e \tag{43a}$$

$$\dot{v} = -\kappa(\eta)(1 + v)v + (1 + v)\,\mathrm{Re}(e\bar{w}) \tag{43b}$$

$$\dot{z} = -\mu(\eta)z + \mathrm{Im}(e\bar{w}) \tag{43c}$$

Notice that for $e = 0$ the system reduces to the one in Eqs. (24).

For α large enough, Eq. (43a) is essentially a boundary-layer subsystem to the slow system given by Eqs. (43b) and (43c). Singular perturbation theory[21] guarantees that, as soon as the error becomes small enough, the (z, v) trajectories of the system will follow the ones of Eqs. (24).

Next we show that the control law in Eq. (41) is well defined in the sense that it remains bounded for all $t \geq 0$. We show that, with α large enough, $w(t) \neq 0$ for all $t \geq 0$, i.e., $w(t)$ tends to zero only asymptotically for all initial conditions inside an a priori given compact set.

Proposition 1: Consider the system in Eqs. (43) and the compact set

$$\mathcal{N}_\beta = \left\{(\omega, w, z) \in \mathcal{W} : |e|[(1 + v)/v]^{\frac{1}{2}} \leq \beta\right\} \tag{44}$$

where $\mathcal{W} = \mathbb{C} \times (\mathbb{C}\backslash\{0\}) \times \mathbb{R}$, and let $\mu_c > \kappa_c > 0$ and $\alpha > (\kappa_c + \beta)/2$. Then for all initial conditions in $\mathcal{N}_\beta$, $|w|$ is bounded below by an exponentially decaying function.

Proof: Equation (43b) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}|w|^2 = -(1 + |w|^2)\,[\kappa(\eta)|w|^2 - \mathrm{Re}(e\bar{w})] \tag{45}$$

Note that, from Eq. (43a), $|e(t)| \leq |e(0)|e^{-\alpha t}$ and, using Eq. (44),

$$|e(t)| \leq \beta\left(\frac{|w(0)|^2}{1 + |w(0)|^2}\right)^{\frac{1}{2}} \exp\left[\frac{-(\kappa_c + \beta)t}{2}\right], \qquad t \geq 0 \tag{46}$$

Consider now the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}|\hat{w}|^2 = -(\kappa_c + \beta)(1 + |\hat{w}|^2)|\hat{w}|^2 \tag{47}$$

The solution of this equation is given by

$$|\hat{w}(t)| = \frac{1}{c_0\exp[(\kappa_c + \beta)t/2] - 1} \geq c_0^{-\frac{1}{2}}\exp\left[\frac{-(\kappa_c + \beta)t}{2}\right] \tag{48}$$

where $c_0 = [|\hat{w}(0)|^2 + 1]/|\hat{w}(0)|^2$. Comparison of Eqs. (46) and (48) implies that

$$|e(t)| \leq \beta|\hat{w}(t)|, \qquad \forall t \geq 0 \tag{49}$$

where $|\hat{w}|$ obeys Eq. (47) with $|\hat{w}(0)| = |w(0)|$.

Notice now that, because $\mathrm{Re}(e\bar{w}) \leq |e||w|$ and using Eq. (49), one has from Eq. (45) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|w|^2 &\geq -(1 + |w|^2)[\kappa(\eta)|w|^2 + |e||w|]\\ &\geq -(1 + |w|^2)[\kappa(\eta)|w|^2 + \beta|\hat{w}||w|]\end{aligned} \tag{50}$$

and because $-\kappa_c \leq \kappa(\eta) \leq \kappa_c$, finally,

$$\frac{\mathrm{d}}{\mathrm{d}t}|w|^2 \geq -(1 + |w|^2)(\kappa_c|w|^2 + \beta|\hat{w}||w|) \tag{51}$$

By comparing Eqs. (47) and (51) and because $|w(0)| = |\hat{w}(0)|$, one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}|w(0)|^2 \geq \frac{\mathrm{d}}{\mathrm{d}t}|\hat{w}(0)|^2 \tag{52}$$

Therefore, there exist some $t^* > 0$ such that $|w(t)| \geq |\hat{w}(t)|$ for all $0 \leq t \leq t^*$. We claim that, actually, $|w(t)| \geq |\hat{w}(t)|$ for all $t \geq 0$, and thus $|w|$ is bounded below by the exponentially decaying function $|\hat{w}|$.

Assume that at some point $t' > 0$ we have that $|w(t')| = |\hat{w}(t')|$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}|w(t')| < \frac{\mathrm{d}}{\mathrm{d}t}|\hat{w}(t')|$$

(see Fig. 5). Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|w(t')|^2 &= -[1 + |w(t')|^2]\big[\kappa_c|w(t')|^2 + \beta|w(t')|^2\big]\\ &= -[1 + |\hat{w}(t')|^2](\kappa_c + \beta)|\hat{w}(t')|^2\\ &= \frac{\mathrm{d}}{\mathrm{d}t}|\hat{w}(t')|^2\end{aligned} \tag{53}$$

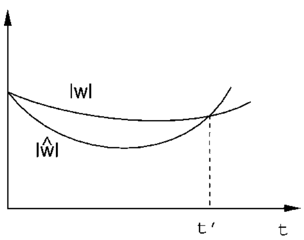

Fig. 5 Time history of $|w|$ and $|\hat{w}|$.

which leads to a contradiction. Therefore, $|w(t)| \geq |\hat{w}(t)|$ and, thus, $w(t) \neq 0$ for all $t \geq 0$. □

In Ref. 10 the control law in Eq. (14) was also implemented using the same methodology. That is, the control for the complete system was given by Eq. (41), where $\kappa = \kappa_c$, $\mu = \mu_c$, and $\dot{\kappa} = \dot{\mu} = 0$. The value of the gain α increases with β, which in turns increases as $|w|$ decreases. That is, when the initial condition is close to $w = 0$, then a faster transient for ω is required. This faster transient is achieved by taking α large enough. A potential problem in the implementation of the control in Eq. (41) is now evident. If e does not decay fast enough so that $\omega \to \omega_d$ sufficiently fast, then there is the danger that w will move toward the z axis before the control law in Eq. (14) becomes effective. This is one more reason that motivated the choice of the control law in Eq. (19). Namely, it is beneficial for w to move away from the z axis. This can reduce the value of the gain α significantly.

In most situations it is not necessary to choose α from Proposition 1. Actually, as the numerical simulations in the next section show, for most practical examples it suffices to choose α to be sufficiently larger than the gains μ_c and κ_c. From Eq. (42) it is also clear that α should be at least as large as $\kappa_c/2$, in order for e/w to remain bounded.

Remark 3: The rigid body problem subject to two control inputs is only but one example of an underactuated mechanical system. Systems of this form can be found in the class of systems subject to nonholonomic, i.e., nonintegrable constraints.[22] Time-invariant control laws for these systems are necessarily nonsmooth, and recently proposed control laws[23–25] include singularities of the same form as in Eq. (14). Therefore, it is conceptually straightforward to extend the results of this paper to this more general case.

VII. Numerical Example

To illustrate the preceding theoretical analysis, we have simulated the differential equations (12) with the two control laws in Eqs. (14) and (19). The gains are chosen as $\kappa_c = 0.5$ and $\mu_c = 2$. The value of the parameter $\rho = 2$. The initial conditions were taken as $w(0) = 0.3 - i0.25$ and $z(0) = 2.5$. The results are shown in Figs. 6 and 7. Figure 6 shows the corresponding closed-loop trajectories, and Fig. 7 shows the magnitude of the angular velocity (control input for the kinematics) $|\omega|$. The solid lines correspond to the new control law in Eq. (19), and the dashed lines correspond to the earlier control law given in Eq. (14). As is evident, there is a substantial decrease in control effort by using the control law in Eq. (19), especially during the initial portion of the trajectory where z is large and $|w|$ is small.

This control law was later implemented through the dynamics in Eq. (12a). A rest-to-rest maneuver was considered; thus, $\omega(0) = 0$. Simulations for several values of α are shown in Figs. 8 and 9. The trajectories in the (z, v) space are very similar to the ones when ω is the control input. In fact, for $\alpha = 10$ the trajectories for the complete system are essentially identical to the ones with control law in Eq. (19). Figure 9 shows that increasing α may increase the control effort, mainly because of the high-gain boundary-layer part of the controller. At any rate, the corresponding control effort for the control law in Ref. 10 is several orders of magnitude higher (not shown here). Moreover, for small α such as $\alpha = 1$ and 4, the control effort for the controller in Ref. 10 is not bounded. In these cases the slow transients of e allowed w to drift toward the z axis before the control law in Eq. (14) is activated. On the other hand, the controller in Eq. (19) forces the system trajectories away from the z axis, thus providing enough time for the dynamic controller to catch up.

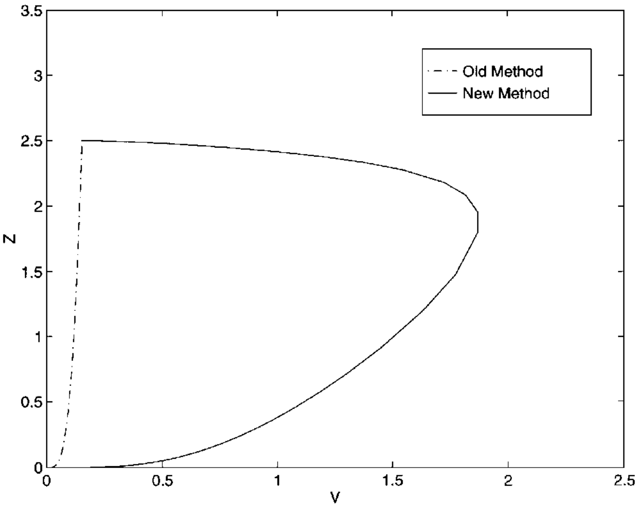

Fig. 6 Closed-loop trajectories for the two methods (kinematics only).

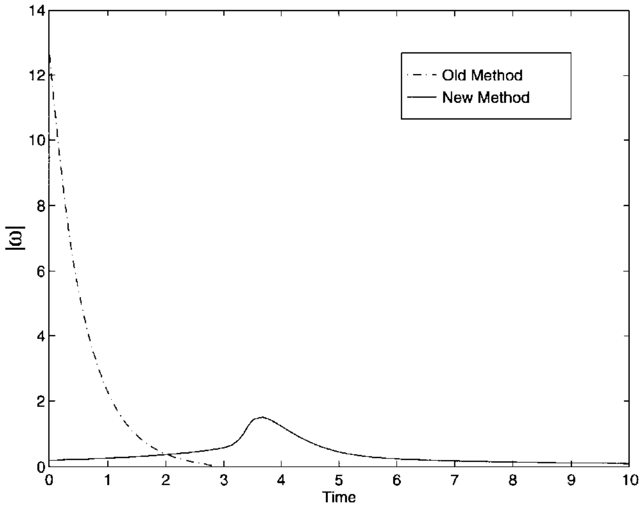

Fig. 7 Control effort for the two methods (kinematics only).

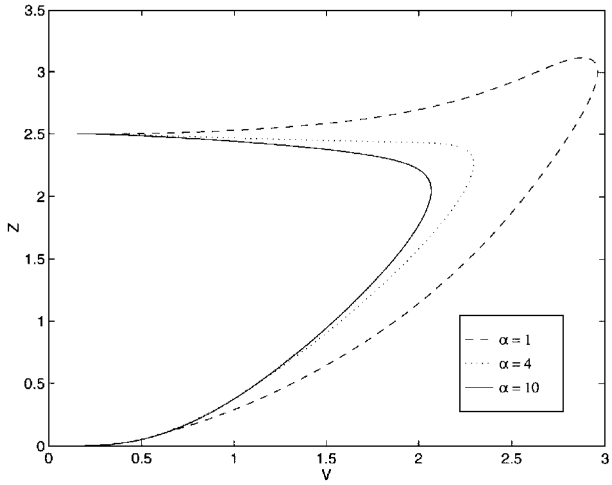

Fig. 8 Closed-loop trajectories for the complete system.

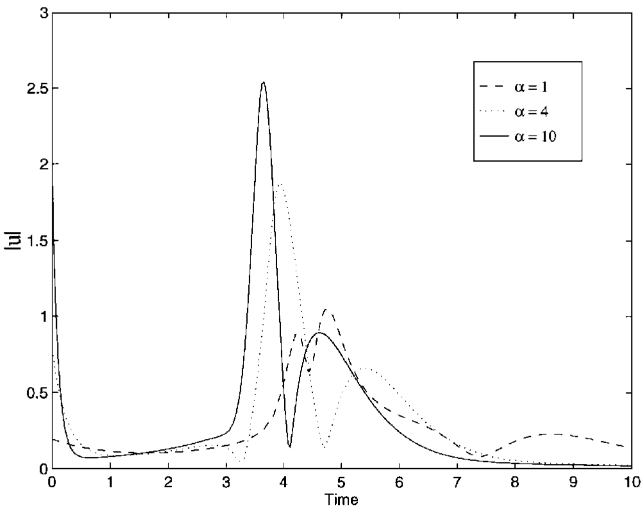

Fig. 9 Control effort for the complete system.

VIII. Conclusions

A nonsmooth control law has been constructed that stabilizes the kinematics of an underactuated rigid spacecraft. It is shown that the proposed control law is well defined and uses considerably less control effort than a previously derived control law. The main idea is to divide the state space into two regions, one that includes initial conditions resulting in high control expenditure and one that includes initial conditions resulting in acceptable control input signals. The proposed control law then forces all of the closed-loop system trajectories to leave the undesirable region of high control effort and, subsequently, use the original control law. Numerical examples indicate a significant control effort reduction using the new control scheme. Because of the limited control torque onboard a spacecraft, for practical situations this may be the difference between feasibility and infeasibility of a particular reorientation maneuver.

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References

[1]Aeyels, D., "Stabilization by Smooth Feedback of the Angular Velocity of a Rigid Body," *Systems and Control Letters*, Vol. 6, No. 1, 1985, pp. 59–63.

[2]Aeyels, D., and Szafranski, M., "Comments on the Stabilizability of the Angular Velocity of a Rigid Body," *Systems and Control Letters*, Vol. 10, No. 1, 1988, pp. 35–39.

[3]Outbib, R., and Sallet, G., "Stabilizability of the Angular Velocity of a Rigid Body Revisited," *Systems and Control Letters*, Vol. 18, No. 2, 1992, pp. 93–98.

[4]Sontag, E., and Sussmann, H., "Further Comments on the Stabilizability of the Angular Velocity of a Rigid Body," *Systems and Control Letters*, Vol. 12, No. 3, 1988, pp. 213–217.

[5]Leonard, N. E., and Krishnaprasad, P. S., "Averaging for Attitude Control and Motion Planning," *Proceedings of the 32nd Conference on Decision and Control* (San Antonio, TX), Inst. of Electrical and Electronics Engineers, Piscataway, NJ, 1993, pp. 3098–3104.

[6]Leonard, N. E., and Krishnaprasad, P. S., "High-Order Averaging on Lie Groups and Control of an Autonomous Underwater Vehicle," *Proceedings of the American Control Conference* (Baltimore, MD), American Automatic Control Council, New York, 1994, pp. 157–162.

[7]Byrnes, C. I., and Isidori, A., "On the Attitude Stabilization of a Rigid Spacecraft," *Automatica*, Vol. 27, No. 1, 1991, pp. 87–95.

[8]Krishnan, H., McClamroch, H., and Reyhanoglu, M., "On the Attitude Stabilization of a Rigid Spacecraft Using Two Control Torques," *Proceedings of the American Control Conference* (Chicago, IL), American Automatic Control Council, New York, 1992, pp. 1990–1995.

[9]Krishnan, H., Reyhanoglu, M., and McClamroch, H., "Attitude Stabilization of a Rigid Spacecraft Using Two Control Torques: A Nonlinear Control Approach Based on the Spacecraft Attitude Dynamics," *Automatica*, Vol. 30, No. 6, 1994, pp. 1023–1027.

[10]Tsiotras, P., Corless, M., and Longuski, M., "A Novel Approach for the Attitude Control of an Axisymmetric Spacecraft Subject to Two Control Torques," *Automatica*, Vol. 31, No. 8, 1995, pp. 1099–1112.

[11]Walsh, G. C., Montgomery, R., and Sastry, S. S., "Orientation Control of a Dynamic Satellite," *Proceedings of the American Control Conference* (Baltimore, MD), American Automatic Control Council, New York, 1994, pp. 138–142.

[12]Morin, P., Samson, C., Pomet, J., and Jiang, Z., "Time-Varying Feedback Stabilization of the Attitude of a Rigid Spacecraft with Two Controls," *Proceedings of the 33rd IEEE Conference on Decision and Control* (Lake Buena Vista, FL), Inst. of Electrical and Electronics Engineers, Piscataway, NJ, 1994, pp. 914, 915.

[13]Coron, J. M., and Kerai, E. L., "Explicit Feedback Stabilizing the Attitude of a Rigid Spacecraft with Two Control Torques," *Automatica*, Vol. 36, No. 5, 1996, pp. 669–677.

[14]Tsiotras, P., and Longuski, J. M., "New Kinematic Relations for the Large Angle Problem in Rigid Body Attitude Dynamics," *Acta Astronautica*, Vol. 32, No. 3, 1994, pp. 181–190.

[15]Tsiotras, P., and Longuski, J. M., "A New Parameterization of the Attitude Kinematics," *Journal of the Astronautical Sciences*, Vol. 43, No. 3, 1995, pp. 243–262.

[16]Morin, P., and Samson, C., "Time-Varying Exponential Stabilization of the Attitude of a Rigid Spacecraft with Two Controls," *Proceedings of the 34th Conference on Decision and Control* (New Orleans, LA), Inst. of Electrical and Electronics Engineers, Piscataway, NJ, 1995, pp. 3988–3993.

[17]Shuster, M. D., "A Survey of Attitude Representations," *Journal of the Astronautical Sciences*, Vol. 41, No. 4, 1993, pp. 439–517.

[18]Longuski, J. M., and Tsiotras, P., "Analytic Solution of the Large Angle Problem in Rigid Body Attitude Dynamics," *Journal of the Astronautical Sciences*, Vol. 43, No. 1, 1995, pp. 25–46.

[19]Tsiotras, P., and Longuski, J. M., "Spin-Axis Stabilization of Symmetric Spacecraft with Two Control Torques," *Systems and Control Letters*, Vol. 23, No. 6, 1994, pp. 395–402.

[20]Brockett, R. W., "Asymptotic Stability and Feedback Stabilization," *Differential Geometric Control Theory*, edited by R. W. Brockett, R. S. Millman, and H. J. Sussman, Birhäuser, Boston, 1983, pp. 181–208.

[21]Khalil, H. K., *Nonlinear Systems*, 2nd ed., Prentice–Hall, Englewood Cliffs, NJ, 1996, p. 380.

[22]Murray, R. M., Li, Z., and Sastry, S. S., *A Mathematical Introduction to Robotic Manipulation*, CRC Press, Boca Raton, FL, 1994, pp. 265–270.

[23]Astolfi, A., "Discontinuous Control of Nonholonomic Systems," *Systems and Control Letters*, Vol. 27, No. 1, 1996, pp. 37–45.

[24]Khennouf, H., and Canudas de Wit, C., "On the Construction of Stabilizing Discontinuous Controllers for Nonholonomic Systems," *Proceedings of IFAC Nonlinear Control Systems Design Symposium* (Tahoe City, CA), International Federation of Automatic Control, New York, 1995, pp. 747–752.

[25]Luo, J., and Tsiotras, P., "Exponentially Convergent Control Laws for n-Dimensional Nonholonomic Systems in Power Form," *Proceedings of the American Control Conference* (Albuquerque, NM), American Automatic Control Council, New York, 1997, pp. 398–402.